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SOME THEOREMS CONCERNING THE NUMBER THEORETICAL FUNCTIONS $\omega(n)$ AND $\Omega(n)$

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Some theorems concerning the number theorectical functions $\omega(n)$ and $\Omega(n)$ *)

J. van de Lune

Abstract

The functions ω and Ω are defined as follows: $\omega(1) = \Omega(1) = 0$ and if $n = p_1^{e_1} p_2^{e_2} \dots p_r^{e_r}$ is the canonical factorization of the natural number n, then $\omega(n) = r$ and $\Omega(n) = e_1 + e_2 + \dots + e_r$. It is known that $\sum_{n\le x}(-1)^{\Omega(n)} = o(x)$, $(x\to\infty)$. There seems to be no corresponding result in the literature for $\omega(n)$. In this report it is shown that $\sum_{n\le x}(-1)^{\omega(n)} = o(x)$, $(x\to\infty)$. Furthermore, it is shown that the series $\sum_{n=1}^{\infty} (-1)^{\omega(n)}/n$ converges to zero. Finally, the remarkable duality relation

$$1 = \sum_{d/n} z^{\omega(d)} (1-z)^{\Omega(\frac{n}{d})} = \sum_{d/r} z^{\Omega(d)} (1-z)^{\omega(\frac{n}{d})}$$

and some of its consequences are discussed.

*) This paper is not for review; it is meant for publication in a journal.

Introduction. As usual, let $\omega(n)$ denote the number of distinct prime divisors and $\Omega(n)$ the total number of prime divisors of the positive integer n. That is, $\omega(1) = \Omega(1) = 0$ and if

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

is the canonical factorization of n, then $\omega(n) = r$ and $\Omega(n) = e_1+e_2+\ldots+e_r$.

It is known [1, p. 123], [4, II, p. 617], [5, p. 74] that

$$\sum_{n \leqq x} (-1)^{\Omega(n)} = o(x), \qquad (x \to \infty).$$

There seems to be no corresponding result in the literature for $\omega(n)$. In this report we will prove that

$$S(x) \overset{\text{def}}{=} \sum_{n \leqq x} (-1)^{\omega(n)} = o(x), \qquad (x \to \infty).$$

We will also establish the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)}}{n}$$

and show that the sum of this series is zero. For the corresponding result for $\Omega(n)$ see [4, pp. 617-621]. The above results can be sharpened considerably, but we will not take the effort here to do so.

Finally we will prove a remarkable duality relation between ω and Ω and discuss some of its consequences.

1. *Proposition 1.1.* The function

$$\phi(s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)}}{n^s},$$

which is obviously analytic for Re $s = \sigma > 1$, has an analytic continuation up to $\sigma \geqq 1$.

Proof. From the definition of $\omega(n)$ it follows immediately that for $\sigma > 1$

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)}}{n^s} = \prod_p \left(1 - \frac{1}{p^s} - \frac{1}{p^{2s}} - \ldots\right), \text{ (p prime).}$$

Thus

$$\zeta(s)\phi(s) = \prod_p \frac{1}{1-\frac{1}{p^s}} \cdot \prod_p \left(1 - \frac{1}{p^s-1}\right) =$$

$$= \prod_p \frac{p^{2s} - 2p^s}{(p^s-1)^2} = \prod_p \left(1 - \frac{1}{(p^s-1)^2}\right) \overset{\text{def}}{=} P(s).$$

Since $p^s - 1$ has all its zeros on the imaginary axis, it follows that $P(s)$ is analytic for $\sigma > \frac{1}{2}$. Furthermore, it is well-known that $\frac{1}{\zeta(s)}$ is analytic for $\sigma \geqq 1$ and it follows that

$$\phi(s) = \frac{1}{\zeta(s)} \cdot P(s)$$

is also analytic for $\sigma \geqq 1$. This completes the proof. □

THEOREM 1.1. $S(x) \overset{\text{def}}{=} \sum_{n \leqq x} (-1)^{\omega(n)} = o(x), \qquad (x \to \infty).$

Before proving this theorem we state the following special version of the well known [2, p. 124] WIENER-IKEHARA Tauberian theorem: Let $F(x)$ be non-negative and non-decreasing for $x \geqq 0$.

Let

$$f(s) = \int_0^{\infty} e^{-sx} F(x)dx$$

converge for $\sigma > 1$. If $f(s)$ is analytic for $\sigma \geqq 1$, except for a simple pole at $s = 1$ with residue A, then

$$\lim_{x\to\infty} \frac{F(x)}{e^x} = A.$$

Proof of Theorem 1.1. Note that

$$1 + (-1)^{\omega(n)} \geqq 0 \text{ for } n = 1,2,3,\ldots .$$

Hence $F(x) \overset{\text{def}}{=} [x] + S(x)$ is a non-negative, non-decreasing function not exceeding $2x$. It follows that for $\sigma > 1$,

$$\zeta(s) + \phi(s) = \sum_{n=1}^{\infty} \frac{1 + (-1)^{\omega(n)}}{n^s} =$$

$$= \sum_{n=1}^{\infty} \frac{F(n) - F(n-1)}{n^s} = \int_{1-0}^{\infty} x^{-s} dF(x) =$$

$$= x^{-s}F(x) \Big|_{1-0}^{\infty} + s \int_1^{\infty} F(x)x^{-s-1} dx =$$

$$= s \int_1^{\infty} F(x)x^{-s-1} dx = s \int_0^{\infty} e^{-sx}F(e^x)dx.$$

Thus

$$\int_0^{\infty} e^{-sx}F(e^x)\,dx = \frac{\zeta(s) + \phi(s)}{s}, \qquad (\sigma > 1).$$

Recall that $\zeta(s)$ is a meromorphic function with only one simple pole at $s = 1$ with residue $A = 1$. From Proposition 1.1 it now follows that the function

$$\frac{\zeta(s) + \phi(s)}{s} - \frac{1}{s-1}$$

is analytic for $\sigma \geqq 1$. Now, applying the WIENER-IKEHARA theorem, we obtain

$$A = 1 = \lim_{x\to\infty} \frac{F(e^x)}{e^x} = \lim_{x\to\infty} \frac{F(x)}{x} =$$

$$= \lim_{x\to\infty} \frac{[x] + S(x)}{x} = 1 + \lim_{x\to\infty} \frac{S(x)}{x}.$$

Thus,

$$\lim_{x\to\infty} \frac{S(x)}{x} = 0 \text{ or } S(x) = o(x)$$

and this proves the theorem. □

THEOREM 1.2. The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)}}{n}$$

converges and its sum is zero.

Before proving this theorem we state the following well known result [1, p. 124].

Let

$$g(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

be absolutely convergent for $\sigma > 0$ and suppose that $g(s)$ is analytic for $\sigma \geqq 0$.

If in addition, $a(n) = o(1)$, then

$$\sum_{n=1}^{\infty} a(n)$$

converges and its sum is $g(0)$.

Proof of Theorem 1.2. Define

$$a(n) = \frac{(-1)^{\omega(n)}}{n}.$$

Then we have

$$g(s) = \sum_{n=1}^{\infty} \frac{(-1)^{\omega(n)}}{n^{s+1}} = \phi(s+1).$$

Since $\phi(s)$ is analytic for $\sigma \geqq 1$, we see that $g(s)$ is analytic for $\sigma \geqq 0$, and it follows that the series appearing in theorem 1.2 converges. Moreover,

$$g(0) = \phi(1) = \lim_{s\to 1} \frac{P(s)}{\zeta(s)} = 0$$

and the theorem is proved. □

Remark. Actually, $\phi(s)$ has a zero of order 2 at $s = 1$ because,

$$\lim_{s\to 1} \frac{\phi(s)}{(s-1)^2} = \lim_{s\to 1} \left\{\frac{1}{(s-1)\zeta(s)} \cdot \frac{1-\frac{1}{(2^s-1)^2}}{(s-1)} \cdot \prod_{p>3} \left(1-\frac{1}{(p^s-1)^2}\right)\right\} =$$

$$= 4 \log 2 \cdot \prod_{p>3} \left(1 - \frac{1}{(p-1)^2}\right) \neq 0.$$

Note by contrast that the corresponding function

$$\sum_{n=1}^{\infty} \frac{(-1)^{\Omega(n)}}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

only has a simple zero at $s = 1$.

2. From their definitions it is hardly to be expected that there is much of a relation between $\omega(n)$ and $\Omega(n)$. However, we will exhibit below a remarkable duality relation between the two functions and discuss some of its consequences.

We first note that [3, p. 355]

$$\omega(n) = O\left(\frac{\log n}{\log\log n}\right),$$

from which it is easily seen that if Re $s = \sigma > 1$ then

$$\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s}$$

is an entire function of z. Also, for $\sigma > 1$ and $|z| < 2^{\sigma}$, it is clear that

$$\sum_{n=1}^{\infty} \frac{z^{\Omega(n)}}{n^s} = \prod_p \left(1 + \frac{z}{p^s} + \frac{z^2}{p^{2s}} + \ldots\right) = \prod_p \frac{1}{1 - \frac{z}{p^s}}$$

is an analytic function of z.

THEOREM 2.1. If $|z| < 2^{\sigma}$ and $\sigma > 1$ then

$$(2.1)\qquad \left(\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{(1-z)^{\Omega(n)}}{n^s}\right) = \zeta(s).$$

Proof.

$$\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{z}{p^s} + \frac{z}{p^{2s}} + \frac{z}{p^{3s}} + \ldots\right) =$$

$$= \prod_p \left(1 + z\,\frac{\frac{1}{p^s}}{1 - \frac{1}{p^s}}\right) = \prod_p \frac{p^s + z - 1}{p^s - 1}$$

and

$$\sum_{n=1}^{\infty} \frac{(1-z)^{\Omega(n)}}{n^s} = \prod_p \left(1 + \frac{1-z}{p^s} + \frac{(1-z)^2}{p^{2s}} + \frac{(1-z)^3}{p^{3s}} + \ldots\right) =$$

$$= \prod_p \left(1 + \frac{\frac{1-z}{p^s}}{1 - \frac{1-z}{p^s}}\right) = \prod_p \frac{p^s}{p^s + z - 1} .$$

Hence

$$\left(\sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s}\right) \cdot \left(\sum_{n=1}^{\infty} \frac{(1-z)^{\Omega(n)}}{n^s}\right) =$$

$$= \prod_p \frac{p^s + z - 1}{p^s - 1} \prod_p \frac{p^s}{p^s + z - 1} = \prod_p \frac{1}{1 - \frac{1}{p^s}} = \zeta(s)$$

and the theorem follows easily. □

Performing Dirichlet multiplication in (2.1), equating coefficients and changing z into 1 - z we obtain:

$$\sum_{d|n} z^{\omega(d)} (1-z)^{\Omega(\frac{n}{d})} = \sum_{d|n} z^{\Omega(d)} (1-z)^{\omega(\frac{n}{d})} = 1,$$

a remarkable duality relation between ω and Ω.

We now study the analytic continuation of

$$g_z(s) \stackrel{def}{=} \sum_{n=1}^{\infty} \frac{z^{\omega(n)}}{n^s}, \text{ (z fixed, } \sigma > 1)$$

as a function of s. We first have

Proposition 2.1. If $|1-z| \leqq 2$ and $\sigma > 1$ then

$$(2.2) \qquad \frac{g_z'(s)}{g_z(s)} = z\frac{\zeta'(s)}{\zeta(s)} + z(z-1)\sum_p \frac{\log p}{(p^s+z-1)(p^s-1)}$$

(where all derivatives are taken with respect to s).

Proof. From (2.1) we obtain

$$g_z(s) \cdot \prod_p \left(1 + \frac{1-z}{p^s} + \frac{(1-z)^2}{p^{2s}} + \ldots\right) = \zeta(s),$$

which can be written as

$$g_z(s) = \zeta(s) \cdot \prod_p \frac{p^s + z - 1}{p^s}.$$

Taking logarithmic derivatives, we find

$$\frac{g_z'(s)}{g_z(s)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_p \left(\frac{p^s}{p^s + z - 1} - 1\right)\log p =$$

$$= \frac{\zeta'(s)}{\zeta(s)} + \sum_p \frac{1 - z}{p^s + z - 1} \log p =$$

$$= \frac{\zeta'(s)}{\zeta(s)} + \sum_p \left(\frac{1 - z}{p^s + z - 1} - \frac{1-z}{p^s-1}\right) \log p + \sum_p \frac{1-z}{p^s-1}\log p =$$

$$= z \cdot \frac{\zeta'(s)}{\zeta(s)} + z(z-1) \sum_p \frac{\log p}{(p^s+z-1)(p^s-1)},$$

which completes the proof. □

If $|1-z| \leq \sqrt{2}$, then

$$R_z(s) \overset{\text{def}}{=} \sum_p \frac{\log p}{(p^s+z-1)(p^s-1)}$$

is regular for $\sigma > \frac{1}{2}$. It then follows that $\frac{g_z'(s)}{g_z(s)}$ is regular for $\sigma > \frac{1}{2}$ except at $s = 1$ and at the (possible) zeros of $\zeta(s)$ situated at the right of the line $\sigma = \frac{1}{2}$.

Integrating the formula

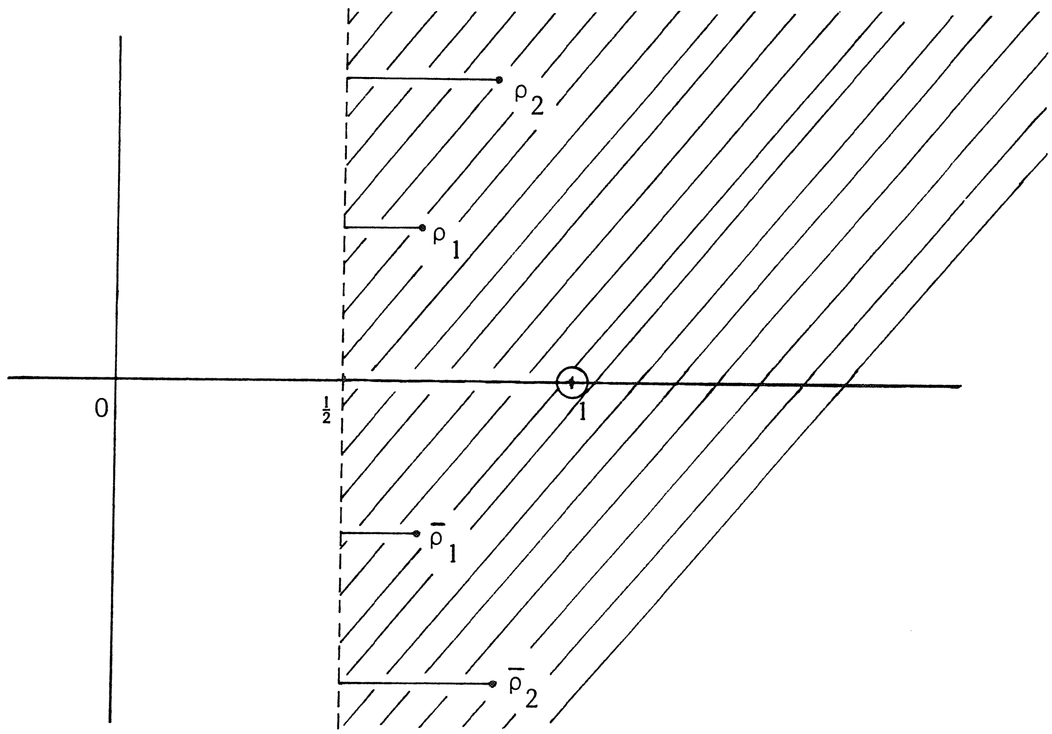
$$\frac{g_z'(s)}{g_z(s)} = z\,\frac{\zeta'(s)}{\zeta(s)} + z(z-1)R_z(s)$$

we get

$$g_z(s) = \zeta^z(s)\,\exp(P_z(s)),$$

where $P_z(s)$ is analytic for $\sigma > \frac{1}{2}$.

Thus $g_z(s)$ is analytic in the shaded region below, where the ρ's stand for the (possible) zeros of $\zeta(s)$ which lie at the right of the line $\sigma = \frac{1}{2}$.



Hence, if (for example) z is irrational and $|1-z| \leqq \sqrt{2}$, we find, surprisingly enough, regardless of which z is chosen subject to the above conditions, that the set of singularities of $g_z(s)$ in the halfplane $\sigma > \frac{1}{2}$ always consists of the same points, namely $s = 1$ and the zeros of $\zeta(s)$ lying in the halfplane $\sigma > \frac{1}{2}$. This seems to lend credence to the Riemann-hypothesis.

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